Addendum: Estimation of Maximal Reliability

The concern of this section is to present a readily applicable procedure for estimation of maximal reliability for a linear combination of congeneric measures, denoted Y_1, Y_2, …, Y_k ($k > 1$; in the case $k = 2$, add identifying restrictions)

$$Y = w_1Y_1 + w_2Y_2 + \ldots + w_kY_k\,,$$

and of their optimal weights w_1, w_2, …, w_k. (For these measures, the well-known classical test theory related decomposition $Y_i = b_i\,\eta + E_i$ holds, where b_i are the indicator loadings on the common true score η and E_i are the error scores with variances θ_i, $i = 1, \ldots, k$). The following method is described in detail in Raykov (in press; see above in this document for information how to download relevant material related to that paper).

As has been well documented in the psychometric literature (e.g., Raykov, in press and references therein), the maximal reliability for a scale of congeneric components is

$$\rho_{max} = \frac{\sum_{i=1}^{k}\frac{\rho_i}{1-\rho_i}}{1+\sum_{i=1}^{k}\frac{\rho_i}{1-\rho_i}}\ ,$$

where ρ_i is the reliability coefficient of Y_i ($i = 1, \ldots, k$; e.g., Li, 1997). This highest reliability is accomplished with the choice of

$$w_i = \frac{\rho_i}{b_i(1-\rho_i)}$$

($i = 1, 2, \ldots, k$) as individual component weights (e.g., Conger, 1980). As an alternative to the possibly too laborious conventional methods utilizing eigenvalue determination and related activities, one can use LISREL 8.54 for Windows with the following nonlinear constraints for the pertinent factor loadings.

$$w_i = b_i \theta_{ii}^{-1}$$

as the optimal component weights (i = 1, …, k; see Raykov, in press, for details on how to obtain this expression).

With this approach, an estimate of maximal reliability ρ_{max} is obtained by estimating the correlation $Corr(\eta, Y)$ between the underlying latent construct η and the optimal linear combination Y ensured by choosing the individual weights w_i as in the last equation (i = 1, …, k). This method also yields automatically standard error estimates for these weights. Further details are provided in Raykov (in press).

The LISREL syntax for accomplishing maximal reliability estimation (and standard error estimation for the optimal weights) is as follows.

```
LISREL INPUT FILE FOR MAXIMAL RELIABILITY AND OPTIMAL WEIGHT
ESTIMATION. THIS FILE REPRESENTS THE SOURCE CODE USED FOR THE
EXAMPLE IN THE ILLUSTRATION SECTION (WITH k = 5 CONGENERIC
MEASURES), AND SHOULD BE ACCORDINGLY MODIFIED WITH A
DIFFERENT NUMBER OF CONGENERIC COMPONENTS.
SEE RAYKOV (IN PRESS) FOR A NUMERICAL EXAMPLE
DA NI=5 NO=<SAMPLE SIZE>
CM=<FILE NAME>
MO NY=5 NE=7 PS=SY,FI BE=FU,FI TE=ZE ! ERROR VARIANCES RELEGATED
LE                                    ! TO PS(I,I), I = 1, …, 5 (SEE BELOW)
T1 T2 T3 T4 T5 ETA COMPOSIT ! COMPOSIT=AUXILIARY VARIABLE (L COM)
VA 1 PS 6 6                           ! FIX LATENT SCALE FOR IDENTIFIABILITY
FR BE 1 6 BE 2 6 BE 3 6 BE 4 6 BE 5 6 ! FREE INDICATOR LOADINGS.
FR BE 7 1 BE 7 2 BE 7 3 BE 7 4 BE 7 5 ! PREPARE FOR CONSTRAINTS BELOW.
```

```
FR PS 1 1 PS 2 2 PS 3 3 PS 4 4 PS 5 5    ! MEASUREMENT ERROR VARIANCES
VA 1 LY 1 1 LY 2 2 LY 3 3 LY 4 4 LY 5 5 ! SET Yi = ηi  (i = 1, …, 5)
CO BE(7,1)=BE(1,6)*PS(1,1)**-1 ! THIS AND NEXT 4 LINES REPRESENT
CO BE(7,2)=BE(2,6)*PS(2,2)**-1 ! THE OPTIMAL WEIGHT CONSTRAINTS IN
CO BE(7,3)=BE(3,6)*PS(3,3)**-1 ! EQUATIONS (5), AND ENSURE THAT
CO BE(7,4)=BE(4,6)*PS(4,4)**-1 ! 'COMPOSIT' = LINEAR COMBINATION OF Y'S
CO BE(7,5)=BE(5,6)*PS(5,5)**-1 ! WITH MAXIMAL RELIABILITY
ST .5 ALL ! THESE START VALUES MAY OR MAY NOT BE NEEDED, OR
OU NS      ! OTHER START VALUES MAY BE NECESSARY, WITH OTHER DATA
```

References

Conger, A. (1980). Maximally reliable composites for undimensional measures. Educational and Psychological Measurement, 40, 367-375.

Li, H. (1997). A unifying expression for the maximal reliability of a linear composite. Psychometrika, 62, 245-249.

Raykov, T. (in press). Estimation of maximal reliability: A note on a covariance structure modeling approach. British Journal of Mathematical and Statistical Psychology.