

Hypothesis Testing for Var-Cov Components

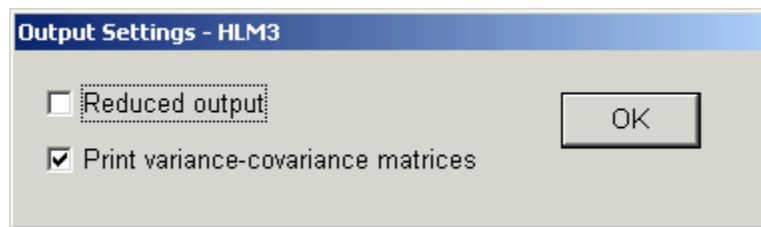
When the specification of coefficients as fixed, random or non-randomly varying is considered, a null hypothesis of the form

$$H_0: \tau_{qq} = 0$$

is considered, where

$$\tau_{qq} = \text{Var}(\beta_{qj}).$$

Additional output files are obtained by selecting the **Output Settings** option from the **Other Settings** menu to bring up the **Output Settings – HLM3** dialog box. On this dialog box, the **Print variance-covariance matrices** option should be checked.



With the above selection, an additional three variance-covariance files will be generated for a HLM3 model along with the output file, these being **gamvc.dat**, **gamvcr.dat** and **tauvc.dat**. **Tauvc.dat** contains the tau matrices and the inverse of the associated information matrix. For a model with r random effects at level-1, and $r2$ random effects at level-2, **tauvc.dat** contains the following:

τ_x , in r columns and r rows

τ_β , in $r2$ columns and $r2$ rows

the inverse of the information matrix. This dimensions of the matrix are

$$\left(\frac{r(r+1)}{2} + \frac{r2(r2+1)}{2} \right) \times \left(\frac{r(r+1)}{2} + \frac{r2(r2+1)}{2} \right)$$

The **tauvc.dat** file for a HLM3 example is given below:

```

0.6404879  0.0467574
0.0467574  0.0112249
0.1653112  0.0170460
0.0170460  0.0110198
0.0006327  0.0000285  0.0000014  -0.0000281  -0.0000014  -0.0000001
0.0000285  0.0000249  0.0000020  -0.0000014  -0.0000011  -0.0000001
0.0000014  0.0000020  0.0000039  -0.0000001  -0.0000001  -0.0000002
-0.0000281  -0.0000014  -0.0000001  0.0013258  0.0001252  0.0000117
-0.0000014  -0.0000011  -0.0000001  0.0001252  0.0000518  0.0000087
-0.0000001  -0.0000001  -0.0000002  0.0000117  0.0000087  0.0000063
    
```

For this model, $r = r2 = 2$, so that $\tau_x : 2 \times 2$, $\tau_\beta : 2 \times 2$, and the information matrix is of dimension 6×6 .

The first two lines contain the estimates of the level-2 variance-covariance matrix, τ_π , and corresponds with the Tau(pi) matrix as listed in the output file. The elements of this matrix are $\tau_{\pi_{00}}$, $\tau_{\pi_{01}}$, $\tau_{\pi_{10}}$, and $\tau_{\pi_{11}}$, with $\tau_{\pi_{00}} = 0.6404879$, $\tau_{\pi_{01}} = \tau_{\pi_{10}} = 0.0467574$, and $\tau_{\pi_{11}} = 0.0112249$ respectively.

The third and fourth lines are the level-3 variance-covariance matrix, τ_β . We see that $\tau_{\beta_{00}} = 0.1653112$, $\tau_{\beta_{01}} = \tau_{\beta_{10}} = 0.0170460$, and $\tau_{\beta_{11}} = 0.0110298$.

This matrix is also called the asymptotic covariance matrix of the estimated β -elements. The matrices are followed by the inverse of the information matrix, \hat{V} . The information matrix can be divided into four blocks. The two blocks on the diagonal (see table below) contain the variances of the elements of the τ matrices at levels 2 and 3 respectively. The off-diagonal blocks, which are the same due to the symmetry of the information matrix, contains the covariances of the elements of the τ matrices.

	$\tau_{\pi_{00}}$	$\tau_{\pi_{01}}$	$\tau_{\pi_{11}}$	$\tau_{\beta_{00}}$	$\tau_{\beta_{01}}$	$\tau_{\beta_{11}}$
$\tau_{\pi_{00}}$	0.0006327	0.0000285	0.0000020	-0.0000281	-0.0000014	-0.0000001
$\tau_{\pi_{01}}$	0.0000285	0.0000249	0.0000020	-0.0000014	-0.0000011	-0.0000001
$\tau_{\pi_{11}}$	0.0000014	0.0000020	0.0000039	-0.0000001	-0.0000001	-0.0000002
$\tau_{\beta_{00}}$	-0.0000281	-0.0000014	-0.0000001	0.0013258	0.0001252	0.000011
$\tau_{\beta_{01}}$	-0.0000014	-0.0000011	-0.0000001	0.0001252	0.0000518	0.0000087
$\tau_{\beta_{11}}$	-0.0000001	-0.0000001	-0.0000002	0.0000117	0.0000087	0.0000063

The square roots of the diagonal elements of this matrix are the standard errors of the τ s. For example, if we denote the q -th diagonal element of the matrix by \hat{V}_{qq} , it follows that \hat{V}_{11} is the square of the standard error of the estimated level-2 variance component $\tau_{\pi_{00}}$, \hat{V}_{22} is that of $\tau_{\pi_{01}}$, and \hat{V}_{33} is that of $\tau_{\pi_{11}}$. The remaining three diagonal elements are the square of the standard errors of the level-3 variances and covariance, the τ_β s. The standard errors of $\tau_{\pi_{00}}$,

$\tau_{\pi_{01}}, \tau_{\pi_{11}}, \tau_{\beta_{00}}, \tau_{\beta_{01}}, \tau_{\beta_{11}}$ are thus 0.025153529, 0.00498999, 0.001974842, 0.036411537, 0.007197222, and 0.00250998 respectively.

From the almost zero values of all the covariances shown in the off-diagonal blocks, it can be concluded that the elements of the estimated τ_x are uncorrelated with the elements of the estimated τ_β .

Based on the information matrix, hypothesis tests and confidence intervals for the individual τ s may be constructed. For example, we can compute the 95% confidence interval for the variance component $\tau_{\pi_{11}}$ as shown below:

$$\begin{aligned} 95\% \text{ C.I. of } \tau_{\pi_{11}} &= \hat{\tau}_{\pi_{11}} \pm 1.96 \times (\hat{V}_{33})^{1/2} \\ &= 0.0112249 \pm 1.96 \times \sqrt{0.0000039} \\ &= 0.0112 \pm 0.0039 \\ &= (0.0081, 0.0151). \end{aligned}$$

In addition, one can test whether significant random variation exists for random coefficient τ_{qq} with a null hypothesis of the form

$$H_0 : \tau_{qq} = 0$$

against the alternative hypothesis

$$H_1 : \tau_{qq} \neq 0,$$

where $\tau_{qq} = \text{Var}(\beta_q)$. Rejection of the null hypothesis will lead to the conclusion that there is random variation in β_q . If most or all groups have sufficient data to compute OLS estimates, this can be tested under the model

$$\beta_q = \gamma_{q0} + \sum_{j=1}^{S_q} \gamma_{qj} W_{sj},$$

using the statistic

$$\sum_j (\hat{\beta}_{qj} - \hat{\gamma}_{q0} - \sum_{s=1}^{S_q} \hat{\gamma}_{qs} W_{sj})^2 / \hat{V}_{qq}$$

which will be distributed approximately χ^2 with $J - S_q - 1$ degrees of freedom (Raudenbush & Bryk, 2002).

A test for this purpose is based on the standard error of $\hat{\tau}_{qq}$, as obtained from the information matrix shown above. The statistic

$$Z = \hat{\tau}_{qq} / \left[\text{var} \left(\hat{\tau}_{qq} \right) \right]^{1/2}$$

is approximately normally distributed under the large sample theory of maximum likelihood estimates.

The assumption underlying both the confidence interval shown above and the single parameter hypothesis test for random variation in β_q depends on the quality of the normal approximation. For large samples and non-normal data, the use of the ratio shown above is preferred. If this test is used to test the significance of a variance component, it is appropriate to use a one-sided test (Hox, 2002).

However, when the normality approximation is rather poor, as will be the case when τ_{qq} is near zero and a small sample of groups, the distribution of this statistic is non-normal as the sampling distribution of the variances is skewed. As pointed out by Raudenbush & Bryk (2002): "In general, the smaller the variance component, the more data are needed to justify the large-sample normality approximation on which this test is based". A possible rule of thumb for evaluating the size of τ_{qq} is to avoid using this statistic if the contribution of τ_{qq} to the total variation in the model is less than 1%.

When use of the statistic discussed above is inadvisable, the chi-square test based on the residuals as used by Raudenbush & Bryk (2002), is the appropriate choice. For a two-level model, this χ^2 is defined as (also see above)

$$\chi^2 = \sum \left(\hat{\beta}_j - \beta \right)^2 / \hat{V}_j$$

with degrees of freedom equal to $J - p - 1$, with J the number of level-2 units, and p is the number of predictors in the model. $\hat{\beta}_j$ is the OLS estimate of a regression coefficient, computed separately in group j , with its estimated sampling variance in group j denoted by \hat{V}_j . This is the approach followed in the calculation of the χ^2 - statistics reported for individual variance components as reported in the HLM output.

For more on this topic, please see:

- Raudenbush, S.W. & Bryk, A.S. (2002). *Hierarchical Linear Models. Applications and Data Analysis Methods*. Second Edition. Sage. pp. 61-64; 283-284.
- Hox, J. (2002). *Multilevel Analysis. Techniques and Applications*. IEA. pp.42-3.